Beta Distribution of Kind 2nd

Let 'X' be a positive continuous random variable with interval $(0,\infty)$ is said to be Beta distribution of 2nd kind, having its p.d.f

$$f(x)=\frac{1}{\beta(a,b)}\frac{x^{a-1}}{(1+x)^{a+b}} \qquad 0\le x\le\infty$$

And its function is

$$\beta(a,b)=\int_0^\infty \frac{x^{a-1}}{(1+x)^{a+b}}dx$$

It has two parameters a & b.

Properties

i) Beta distribution is a continuous distribution.
ii) The total area under the curve is unity.
iii) The range of the distribution is 0 to ∞.
iv) It has two parameters a & b.

v) The mean of the beta distribution of 2nd kind is $\mathrm{E(x)}=\frac{a}{b-1}$.

vi) The variance of the beta distribution of 2nd kind is $\mathrm{Var(x)}=\frac{a}{b-1}\left[\frac{a+b-1}{(b-1)(b-2)}\right]$.

vii) The mode of the beta distribution of 2nd kind is $Mode=\frac{a-1}{b+1}$

viii) The harmonic mean of the beta distribution of 2nd kind is $H.M=\frac{a-1}{b}$.

Prove that total area under the curve is unity

Proof:
Let by definition
Total Area: Area= $\int f(x)dx$
As x ≈ beta 2nd (a,b)

$$f(x)=\frac{1}{\beta(a,b)}\frac{x^{a-1}}{(1+x)^{a+b}} \qquad 0\le x\le\infty$$

$$Area=\int_0^\infty\frac{1}{\beta(a,b)}\frac{x^{a-1}}{(1+x)^{b+1}}dx$$

$$Area=\frac{1}{\beta(a,b)}\int_0^\infty\frac{x^{a-1}}{(1+x)^{b+1}}dx \qquad (A)$$

As we know that beta function is

$$\beta(a,b)=\int_0^\infty \frac{x^{a-1}}{(1+x)^{a+b}}dx \qquad (B)$$

Comparing (A) & (B) and we get
a = a & b = b

$$\beta(a,b)=\beta(a,b)$$

Put in (A)

Area $=\frac{1}{\beta(a,b)}\beta(a,b)=1$

Hence Proved

Derive rth moment about origin and use it to find mean & variance

Solution: Let by definition

$$\mu_r' = E\left(x^r\right) = \int x^r f(x)dx$$

As x ≈ beta 2nd (a,b)

$$f(x) = \frac{1}{\beta(a,b)} \frac{x^{a-1}}{(1+x)^{a+b}} \qquad 0 \le x \le \infty$$

$$\mu_r' = \int_0^\infty x^r \frac{1}{\beta(a,b)} \frac{x^{a-1}}{(1+x)^{a+b}} dx$$

$$\mu_r' = \frac{1}{\beta(a,b)} \int_0^\infty \frac{x^{(r+a)-1}}{(1+x)^{a+b+r-r}} dx$$

$$\mu_r' = \frac{1}{\beta(a,b)} \int_0^\infty \frac{x^{(r+a)-1}}{(1+x)^{(a+r)+(b-r)}} dx \qquad \text{(A)}$$

As we know that beta function is

$$\beta(a,b) = \int_0^\infty \frac{x^{a-1}}{(1+x)^{a+b}} dx \qquad \text{(B)}$$

Comparing (A) & (B) and we get

a = a+r & b = b-r

$$\beta(a,b) = \beta(a+r, b-r)$$ Put in (A)

$$\mu_r' = \frac{1}{\beta(a,b)} \beta(a+r, b-r)$$

$$\mu_r' = \frac{\dfrac{\Gamma(r+a)\Gamma(b-r)}{\Gamma(r+a+b-r)}}{\dfrac{\Gamma(a)\Gamma(b)}{\cdot\Gamma(a+b)}}$$

$$\mu_r' = \frac{\Gamma(r+a)\Gamma(b-r)}{.\Gamma(a)\Gamma(b)}$$

$$\mu_r' = \frac{\Gamma(r+a)\Gamma(b-r)}{\Gamma(a)\Gamma(b)} \qquad \text{©}$$

Hence the required result

Mean = $\mu_1' = E(x)$ Put r = 1 in eq (C)

$$\mu_1' = \frac{\Gamma(1+a)\Gamma(b-1)}{\Gamma(a)\Gamma(b)}$$

$$= \frac{a\Gamma(a)\Gamma(b-1)}{\Gamma(a)(b-1)\Gamma(b-1)}$$

$$\mu_1' = \frac{a}{b-1}$$

Now, put r = 2 in eq.(C)

$$\mu_2' = \frac{\Gamma(2+a)\Gamma(b-2)}{\Gamma(a)\Gamma(b)}$$

$$\mu_2' = \frac{(a+1)\Gamma(a+1)\Gamma(b-2)}{\Gamma(a)(b-1)\Gamma(b-1)}$$

$$\mu_2' = \frac{a(a+1)\Gamma(a)\Gamma(b-2)}{\Gamma(a)(b-1)(b-2)\Gamma(b-2)}$$

$$\mu_2' = \frac{a(a+1)}{(b-1)(b-2)}$$

$$\mu_2 = Var(X) = E(X^2) - [E(X)]^2 = \mu_2' - (\mu_1')^2$$

$$Var(x) = \frac{a(a+1)}{(b-1)(b-2)} - \left(\frac{a}{b-1}\right)^2$$

$$\text{Var(X)} = \frac{a}{b-1}\left[\frac{(a+1)}{(b-2)} - \left(\frac{a}{b-1}\right)\right]$$

$$Var(x) = \frac{a}{b-1}\left[\frac{(a+1)(b-1) - a(b-2)}{(b-2)}\right]$$

$$Var(x) = \frac{a}{b-1}\left[\frac{ab - a + b - 1 - ab + 2a)}{(b-1)(b-2)}\right]$$

$$Var(x) = \frac{a}{b-1}\left[\frac{a+b-1}{(b-1)(b-2)}\right]$$

Find mode of beta distribution of kind 2nd

Solution:

As x $\approx$ beta 2nd (a,b)

$$f(x) = \frac{1}{\beta(a,b)}\frac{x^{a-1}}{(1+x)^{a+b}} \qquad\qquad 0 \le x \le \infty$$

Let by definition

If following two conditions are satisfied then mode exists.

$$f(x') = 0 \qquad \text{or} \qquad \frac{d}{dx}\log f(x) = 0$$

$$f(x') < 0 \qquad \text{or} \qquad \frac{d^2}{dx^2}\log f(x) < 0$$

$$f(x) = \frac{1}{\beta(a,b)}\frac{x^{a-1}}{(1+x)^{a+b}}$$

Taking log on both sides:

$$\log f(x) = \log \beta(a,b) + (a-1)\log x - (a+b)\log(1+x)$$

Differentiate w.r.t to ‘x’ & equating to zero::

$$\frac{d}{dx}\log f(x) = 0 + \frac{a-1}{x} - \frac{a+b}{1+x}$$

$$\frac{d}{dx}\log f(x) = \frac{(a-1)}{x} - \frac{(a+b)}{(1+x)} \qquad \text{(A)}$$

$$0 = \frac{(a-1)(1+x) - x(a+b)}{x(1+x)}$$

$$x(a+b) = (a-1)(1+x)$$

$$xb + xa = a + ax - 1 - x$$

$$xb + x = a - 1$$

$$x(b+1) = a - 1$$

$$x = \frac{a-1}{(b+1)}$$

Again diff. eq(i) w.r.t to 'x'

$$\frac{d^2}{dx^2}\log f(x) = -\frac{(a-1)}{x^2} + \frac{(a+b)}{(1+x)^2}$$

Put $x = \frac{a-1}{(b+1)}$

$$\frac{d^2}{dx^2}\log f(x) = -\frac{(a-1)}{\left(\frac{a-1}{b+1}\right)^2} + \frac{(a+b)}{(1+\frac{a-1}{b+1})^2}$$

$$\frac{d^2}{dx^2}\log f(x) = -\frac{(a-1)(b+1)^2}{(a-1)^2} + \frac{(a+b)(b+1)^2}{(b+1+a-1)^2}$$

$$\frac{d^2}{dx^2}\log f(x) = -\frac{(a-1)(b+1)^2}{(a-1)^2} + \frac{(a+b)(b+1)^2}{(a+b)^2}$$

$$\frac{d^2}{dx^2}\log f(x) = -(b+1)^2\left[\frac{(a-1)}{(a-1)^2} + \frac{(a+b)}{(a+b)^2}\right] < 0$$

Therefore, $\hat{x} = \frac{a-1}{(b+1)}$ is the required mode of beta distribution of kind 2nd

Find Harmonic mean of beta distribution of kind 2nd.

Solution: Let by definition

H.M = $\dfrac{1}{E\left(\frac{1}{x}\right)}$ (A)

$$E\left(\frac{1}{x}\right) = \int \frac{1}{x} f(x)dx$$

As x ≈ beta 2nd (a,b)

$$f(x) = \frac{1}{\beta(a,b)}\frac{x^{a-1}}{(1+x)^{a+b}} \qquad 0 \le x \le \infty$$

$$E\left(\frac{1}{x}\right) = \frac{1}{\beta(a.b)}\int_0^\infty \frac{1}{x}\frac{x^{a-1}}{(1+x)^{a+b}}dx$$

$$E\left(\frac{1}{x}\right) = \frac{1}{\beta(a.b)}\int_0^\infty \frac{x^{(a-1)-1}}{(1+x)^{(a-1)+(b+1)}}dx \qquad \text{(B)}$$

As we know that beta function is

$$\beta(a,b) = \int_0^\infty \frac{x^{a-1}}{(1+x)^{a+b}}\,dx \qquad \text{(C)}$$

Comparing (B) & (C) and we get

a = a-1 & b = b+1

$$\beta(a,b) = \beta(a-1,b+1) \qquad \text{Put in (B)}$$

$$E\left(\frac{1}{x}\right) = \frac{1}{\beta(a,b)}\beta(a-1,b+1)$$

$$E\left(\frac{1}{x}\right) = \frac{1}{\frac{\Gamma a\,\Gamma b}{\Gamma(a+b)}}\frac{\Gamma(a-1)\Gamma(b+1)}{\Gamma(a-1+b+1)}$$

$$E\left(\frac{1}{x}\right) = \frac{\Gamma(a+b)}{\Gamma a\,\Gamma b}\frac{b\,\Gamma b\,\Gamma(a-1)}{\Gamma(a+b)}$$

$$E\left(\frac{1}{x}\right) = \frac{b\,\Gamma(a-1)}{a-1\,\Gamma(a-1)}$$

$$E\left(\frac{1}{x}\right) = \frac{b}{a-1}$$

Put in (A)

$$\text{H.M} = \frac{1}{\frac{b}{(a-1)}}$$

$$\text{H.M} = \frac{(a-1)}{b}$$

Hence the required harmonic mean of beta distribution of kind 2^{nd}

Derive the moment generating function of beta distribution of 2^{nd} kind

Let by definition of m.g.f of beta distribution of 2^{nd} kind

$$M_0(t) = E(e^{tx}) = \int e^{tx} f(x)dx$$

As we know that

$$e^{tx} = 1 + \frac{(\text{tx})^1}{1!} + \frac{(\text{tx})^2}{2!} + \frac{(\text{tx})^3}{3!} + \dots$$

$$e^{tx} = \sum_{r=0}^{\infty}\frac{(\text{tx})^r}{\text{r!}}$$

$$e^{tx} = \sum_{r=0}^{\infty}\frac{\text{t}^r\text{x}^r}{\text{r!}}$$

Then we get

$$M_0(t) = \int\sum_{r=0}^{\infty}\frac{\text{t}^r\text{x}^r}{\text{r!}} f(x)dx$$

$$M_0(t) = \sum_{r=0}^{\infty}\frac{\text{t}^r}{\text{r!}}\int \text{x}^r\ f(x)dx \qquad \text{(A)}$$

Now we consider

$$\int \text{x}^r\ f(x)dx$$

$$\int x^r \, f(x)dx = \int_0^\infty x^r \frac{1}{\beta(a,b)} \frac{x^{a-1}}{(1+x)^{a+b}} dx$$

$$\int x^r \, f(x)dx = \frac{1}{\beta(a,b)} \int_0^\infty \frac{x^{(r+a)-1}}{(1+x)^{a+b+r-r}} dx$$

$$\int x^r \, f(x)dx = \frac{1}{\beta(a,b)} \int_0^\infty \frac{x^{(r+a)-1}}{(1+x)^{(a+r)+(b-r)}} dx \qquad \text{(B)}$$

As we know that beta function is

$$\beta(a,b) = \int_0^\infty \frac{x^{a-1}}{(1+x)^{a+b}} dx \qquad \text{(C)}$$

Comparing (B) & (C) and we get

a = a+r & b = b-r

$$\beta(a,b) = \beta(a+r, b-r) \qquad \text{Put in (B)}$$

$$\int x^r \, f(x)dx = \frac{1}{\beta(a,b)} \beta(a+r, b-r)$$

$$\int x^r \, f(x)dx = \frac{\dfrac{\Gamma(r+a)\Gamma(b-r)}{\Gamma(r+a+b-r)}}{\dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}}$$

$$\int x^r \, f(x)dx = \frac{\Gamma(r+a)\Gamma(b-r)}{\Gamma(a)\Gamma(b)}$$

$$\int x^r \, f(x)dx = \frac{\Gamma(r+a)\Gamma(b-r)}{\Gamma(a)\Gamma(b)} \qquad \text{(A)}$$

$$M_0(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \frac{\Gamma(r+a)\Gamma(b-r)}{\Gamma(a)\Gamma(b)} \qquad \text{Required m.g.f} \qquad \text{(D)}$$

As we know the relationship b/w m.g.f and rth moment about origin

$$M_0(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} u_r' \qquad \text{(E)}$$

Comparing (D) and (E) we get

$$u_r' = \frac{\Gamma(r+a)\Gamma(b-r)}{\Gamma(a)\Gamma(b)}$$

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